Bivariate Distribution Enumeration

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Statistics on Inversion Sequences

Let $I_n = \{e = (e_1, \cdots, e_n) \mid e_i \leq i - 1\}$ be the set of all inversion sequences of size n.

Definition

$$\mathrm{rlmin}(e) = |\{i \mid e_i < e_j \text{ for all } j > i\}|$$

Definition

$$\mathrm{wrlmin}(e) = |\{i \mid e_i \leq e_j \text{ for all } j > i\}|$$

Definition

Let the Inversion Top Positions be

$$\mathrm{itp}(e) = |\{i \mid e_i > e_j \text{ for some } j > i\}|$$

Triangle 1

$$A(x,y) = \sum_{e \in I_n} x^{\mathrm{size}(e)} y^{\mathrm{wrlmin}(e)} = \sum_{n,k=1}^{\infty} a_{n,k} x^n y^k$$

e	$\mathrm{WRLmin}(e)$	$\mathrm{ITP}(e)$		1	2	3	4	5	6
000	1 2 3		1	1					
001	1 2 3		2		2				
002	1 2 3		3		1	5			
010	1 3	2	4		2	8	14		
011	1 2 3		5		6	26	46	42	
012	1 2 3		6		24	112	220	232	132

Triangle 2

$$B(x,y) = \sum_{e \in I_n} x^{\mathrm{size}(e)} y^{\mathrm{itp}(e)} = \sum_{n,k=1}^{\infty} b_{n,k} x^n y^k$$

e	$\mathrm{WRLmin}(e)$	$\mathrm{ITP}(e)$		0	1	2	3	4
000	1 2 3		1	1				
001	1 2 3		2	2				
002	1 2 3		3	5	1			
010	1 3	2	4	14	8	2		
011	1 2 3		5	42	46	26	6	
012	1 2 3		6	132	232	220	112	24

Statistic on Permutations

Definition

$$\mathrm{stat}(\pi) = |\{\pi_i \mid \text{there exists } j, k \text{ with } i < j < k \text{ and } \pi_j < \pi_i < \pi_k\}|$$

In other words, 2s of 213 patterns.

Triangle 2 Again

$$B(x,y) = \sum_{\pi \in S_n} x^{\text{size}(\pi)} y^{\text{stat}(\pi)} = \sum_{n,k=1}^{\infty} b_{n,k} x^n y^k$$

π		$Stat(e)$		0	1	2	3	4
123	1 2 3		1	1				
132	1 2 3		2	2				
231	1 2 3		3	5	1			
213	1 3	2	4	14	8	2		
312	1 2 3		5	42	46	26	6	
321	1 2 3		6	132	232	220	112	24

This appears as OEIS A145879. Defined as the bivariate distribution of 2s of 321 patterns. Peter Bala conjectures that:

Conjecture

$B(x, y)$ *satisfies*

$$x^2 y \frac{dB}{dx} = -1 + (1 - xy)B - x(1 - y)B^2$$

Question

Which of the analogous statistics are Wilf equivalent?

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Which of the analogous statistics are D-finite or algebraic?